RiNNAL-X: Low-Rank ALM for Structured SDP Relaxations

Di Hou

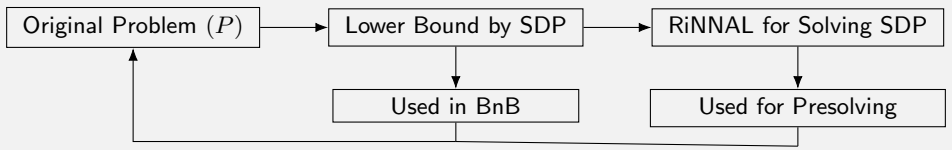
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RiNNAL-X

Common Workflow



Problem	**SDP Relaxation**	**Solver**	**arXiv**
Mixed-Binary QP	DNN	RiNNAL	2502.13849
Mixed-Binary QP	SDP–RLT	RiNNAL+	2507.13776
Polynomial OPT	Poly–SDP / Mom–SOS	RiNNAL–POP	2512.06359
Sparse QP	SC–SDP	RiNNAL–S	2606.22894
Retraction Step	Affine-Constrained Prox	Convex Dual	2602.23626

RiNNAL+

Low-Rank ALM for SDP-RLT Relaxations of MBQP

Problem Formulation

Binary Quadratic Program

$$\begin{aligned} \min_{x} \quad & x^\top Q x + 2c^\top x \\ \text{s.t.} \quad & Ax = b, \quad Bx \le d, \quad x \in \{0,1\}^n. \end{aligned} \tag{BQP}$$

- Extension: mixed-binary variables, complementarity constraints, QCQP.
- SDP relaxation: introduce $X = xx^\top$ and relax it to $X \succeq xx^\top$:

SDP Relaxation

$$\begin{aligned} \min_{Y,\, X,\, x} \quad & \langle \bar{Q}, Y \rangle := \langle Q, X \rangle + 2c^\top x \\ \text{s.t.} \quad & Ax = b, \quad Bx \le d, \quad \operatorname{diag}(X) - x = 0, \\ & Y = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0. \end{aligned}$$

- Lower bound too loose in practice: can be unbounded below.

SDP-RLT Relaxation

Define

$$\bar{A} := \begin{bmatrix} b^\top \\ -A^\top \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} 1 & d^\top & e^\top & 0 \\ 0 & -B^\top & -I_n & I_n \end{bmatrix}.$$

SDP-RLT Relaxation

$$\begin{aligned} \min_{Y,\, X,\, x} \quad & \langle \bar{Q}, Y \rangle \\ \text{s.t.} \quad & \bar{A}^\top Y = 0, \quad \bar{B}^\top Y \bar{B} \geq 0, \quad \operatorname{diag}(X) - x = 0, \\ & Y = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \succeq 0. \end{aligned} \tag{SDP-RLT}$$

Relaxation	Size	Speed	Bound	RelGap
SDP	$\Omega(n^2)$	Fast	Loose	9.424
DNN	$\Omega((n+\ell)^2)$	Slow	Tight	0.836
SDP-RLT	$\Omega(n^2)$	Fast	Tight	0.836

Equivalent

Bottleneck to Solve SDP-RLT

Existing Approaches

- **IPM:** expensive $O(n^6)$ per-iteration cost.
- **ADMM/ALM:** expensive $O(n^3)$ PSD projections/eig. decompositions.

Bottleneck	**Scale**	**RiNNAL+ idea**
PSD variable	$\Omega(n^2)$	low-rank factorization $Y = \widehat{R}\widehat{R}^\top$
Equality constraints	$\Omega(nm)$	structured projection/retraction
Inequality constraints	$\Omega((2n+\ell)^2)$	structured mat-vec/projection
Slater condition fails	structural	work on the exposed face

n: dimension; m: equality constraints; ℓ: inequality constraints.

ALM for (SDP-RLT)

Equality constraints: $\mathcal{E} := \{Y \mid Y_{11} = 1,\ \bar{A}^\top Y = 0,\ \mathrm{diag}(X) - x = 0\}$.

Compact Reformulation

$$\begin{aligned} \min_{Y} \quad & \langle \bar{Q}, Y \rangle + \delta_{\mathcal{E} \cap \mathbb{S}^{n+1}_+}(Y) \\ \text{s.t.} \quad & \bar{B}^\top Y \bar{B} \geq 0 \end{aligned} \tag{SDP-RLT}$$

AL function: $\phi_k(Y) := \langle \bar{Q}, Y \rangle + \dfrac{1}{2\sigma_k}\Big(\big\| [\Lambda^k - \sigma_k \bar{B}^\top Y \bar{B}]_+ \big\|^2 - \|\Lambda^k\|^2 \Big)$.

Augmented Lagrangian Method

$$\begin{aligned} Y^{k+1} &\approx \operatorname*{argmin}_{Y \in \mathcal{E} \cap \mathbb{S}^{n+1}_+} \phi_k(Y), \\ \Lambda^{k+1} &= \big[\Lambda^k - \sigma_k \bar{B}^\top Y^{k+1} \bar{B}\big]_+ . \end{aligned} \tag{ALM-sub}$$

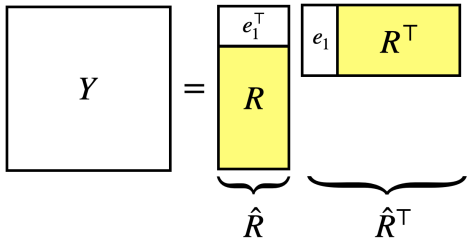
Low-Rank ALM Subproblem

Burer–Monteiro Factorization

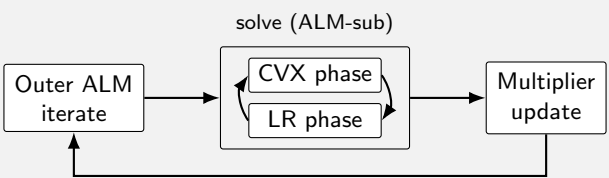
Replace the PSD variable by

$$Y = \widehat{R}\widehat{R}^\top, \qquad \widehat{R} = \begin{bmatrix} e_1^\top \\ R \end{bmatrix}, \quad R \in \mathbb{R}^{n \times r}.$$

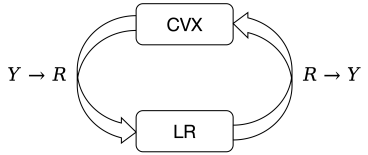
- Dimension: $O(n^2) \to O(nr)$.
- Equivalent if $r \geq \operatorname{rank}(Y^*)$.



RiNNAL+ Framework



A Hybrid Method to Solve ALM Subproblem



CVX Phase

Projected-gradient step on Y

$$\begin{aligned} \min_{Y} \quad & \phi_k(Y) \\ \text{s.t.} \quad & Y \in \mathcal{E} \cap \mathbb{S}_+^{n+1}. \end{aligned}$$

- Global convergence (Lee et al., 2024).
- Efficient PG computation by SSN.
- Automatic rank selection.

LR Phase

Riemannian gradient step on R

$$\begin{aligned} \min_{R} \quad & \phi_k(\widehat{R}\widehat{R}^\top) \\ \text{s.t.} \quad & \widehat{R}\widehat{R}^\top \in \mathcal{E}. \end{aligned}$$

- Smooth manifold by perturbation.
- Efficient projection by Schur and SMW.
- Efficient retraction by Weiszfeld.

CVX Phase: PG Computation

Projected-Gradient Step

$$\begin{aligned} G &= Y - t\nabla\phi(Y), \\ Y^+ &= \Pi_{\mathcal{E}\cap\mathbb{S}^{n+1}_+}(G)\,. \end{aligned}$$

Projection Subproblem

Generalized nearest correlation matrix problem

$$Y^+ = \operatorname{argmin}\left\{\tfrac{1}{2}\|Y-G\|^2 :\ \operatorname{diag}(Y) = e,\ \langle NN^\top, Y\rangle = 0,\ Y \succeq 0\right\}, \qquad \text{(Proj-sub)}$$

efficiently solved via Semismooth Newton-CG method (Qi and Sun, 2006).

Acceleration (10+ times faster)

- Preprocessing: transform $\operatorname{diag}(X) = x$ to $\operatorname{diag}(Y) = e$.
- Warm-start: recover dual variables of (Proj-sub) from LR phase.

CVX Phase: Automatic Rank Selection

- Each CVX projected-gradient step updates Y, hence its numerical rank.
- Use the current $\mathrm{rank}(Y)$ as the initial LR rank; keep the rank fixed within the LR phase.

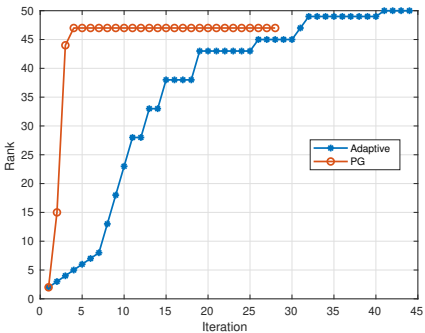


Figure: Initial rank: $r = 2$.

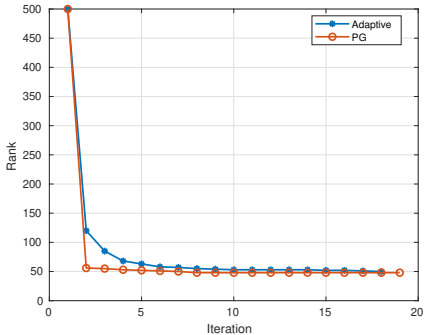


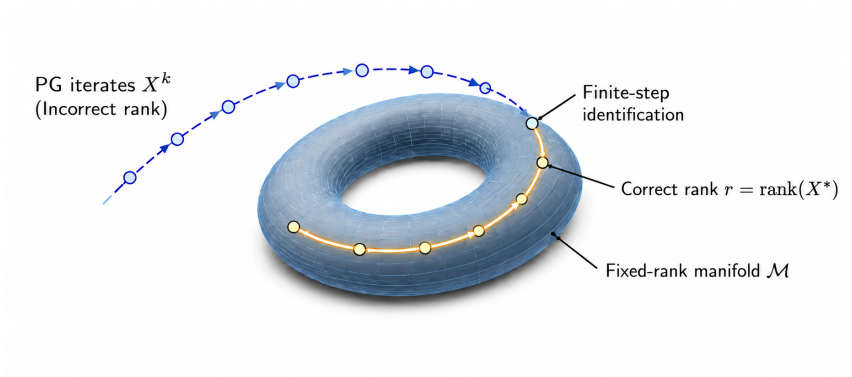
Figure: Initial rank: $r = n$.

CVX Phase: Manifold Identification

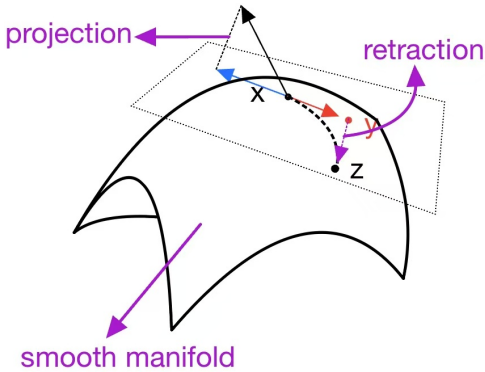
Theorem 1 (Rank Identification)

After finitely many iterations,

$$\operatorname{rank}(Y^k) = \operatorname{rank}(Y^\star) \quad \textit{for all sufficiently large } k.$$



LR Phase: Riemannian Optimization



Riemannian gradient: $R^+ = \mathrm{Retr}_R\big(-\eta\ \mathrm{Proj}_{\mathrm{T}_R\,\mathcal{M}_r}\left(\nabla f_r(R)\right)\big)\,.$

LR Phase: Feasible Region and Smoothness

Theorem 2 (Feasible Region Equivalence)

$$\widehat{R}\widehat{R}^\top \in \mathcal{E} \cap \mathbb{S}_+^{n+1} \quad \Longleftrightarrow \quad R \in \mathcal{M}_r,$$

where

$$\mathcal{M}_r := \left\{ R \in \mathbb{R}^{n\times r} : \begin{array}{ll} AR = be_1^\top & (\textit{affine}), \\ \operatorname{diag}(RR^\top) = Re_1 & (\textit{ball}) \end{array} \right\}.$$

Theorem 3 (Perturbation for Smoothness)

$$\mathcal{M}_{r,v} \;=\; \Big\{ R \in \mathbb{R}^{n\times r} :\; AR = be_1^\top,\; \operatorname{diag}_B\big((2R - ee_1^\top)(2R - ee_1^\top)^\top\big) = e + v \Big\},$$

for a generic random $v \in \mathbb{R}^p$ with $\|v\| = \varepsilon > 0$, every point of $\mathcal{M}_{r,v}$ satisfies LICQ.

LR Phase: Projection

$$h_R(H) := (AH;\ 2\,\mathrm{diag}_B(HR^\top) - H_B e_1).$$

Projection onto $\mathrm{T}_R\,\mathcal{M}_r$

$$\mathrm{grad}\, f_r(R) = \mathrm{Proj}_{\mathcal{T}_R\mathcal{M}_r}(\nabla f_r(R)) = \nabla f_r(R) - h_R^*(\lambda,\mu),$$
$$(\lambda,\mu):\ \text{solution of}\quad h_R(h_R^*(\lambda,\mu)) = h_R(\nabla f_r(R)). \qquad \text{(Proj-sub)}$$

Schur/SMW Reduction

Solve (Proj-sub) by Schur complement and SMW:

$$\mathcal{O}\left(\min\left\{|B|^3 + m^2 r + mr|B|, (mr)^2|B| + (mr)^3\right\}\right),$$

instead of vanilla $\mathcal{O}((|B| + mr)^3)$.

LR Phase: Retraction

Projection Retraction

$$\mathrm{Rtr}_{\bar{R}}(H) := \mathrm{Proj}_{\mathcal{M}_r}(V) = \arg\min \left\{ \|R - V\|_F^2 : \ R \in \mathcal{M}_r \right\},$$

where $\bar{R} \in \mathcal{M}_r$, $H \in \mathcal{T}_{\bar{R}}\mathcal{M}_r$ and $V := \bar{R} + H$.

$$\mathcal{B}_r := \left\{ R \in \mathbb{R}^{n \times r} : \mathrm{diag}_B(RR^\top) - R_B e_1 = 0 \right\}.$$

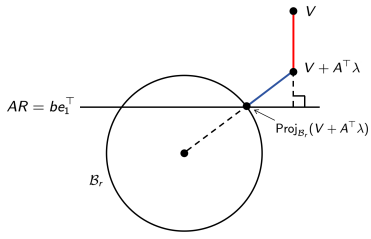
Lemma 4 (Reduction Lemma)

For any $V \in \mathbb{R}^{n \times r}$, if there exists $\lambda \in \mathbb{R}^{m \times r}$ such that

$$A \, \mathrm{Proj}_{\mathcal{B}_r}(V + A^\top \lambda) = b e_1^\top,$$

then

$$\mathrm{Proj}_{\mathcal{M}_r}(V) = \mathrm{Proj}_{\mathcal{B}_r}(V + A^\top \lambda).$$



Retraction Subproblem

Key point: the nonconvex projection problem is reduced to a convex nonsmooth problem in λ.

Convex Subproblem: Generalized Geometric Median

$$\min_{\lambda \in \mathbb{R}^{m \times r}} \sum_{i \in B} \|(V' + A^\top \lambda)_i\| + \sum_{i \in [n] \setminus B} \|(V' + A^\top \lambda)_i\|^2 + \langle b', A^\top \lambda \rangle,$$

where $V' = V - ee_1^\top/2$, and $b' \in \mathbb{R}^{m \times r}$ satisfies $(Ae - 2b)e_1^\top = Ab'$.

Generalized Weiszfeld Algorithm

$$\lambda^{k+1} = T(\lambda^k) := -(A \operatorname{diag}(\mu^k) A^\top)^{-1} A(b' + \operatorname{diag}(\mu^k) V'),$$

where $\mu_i^k = 1/\|(V' + A^\top \lambda^k)_i\|$ for $i \in B$, and $\mu_i^k = 2$ otherwise.

- Convergence guarantee for generalized Weiszfeld.
- Newton acceleration.

Retraction Extension: Affine-Constrained Prox

General Proximal Mapping (Hou et al., 2026)

$$\min_x \left\{ \lambda f(x) + \frac{1}{2}\|x - z\|^2 \ : \ Ax = b \right\}. \tag{P}$$

Theorem (Local Strong Duality)

Under prox-regularity and a nonconvex subdifferential sum rule, for z near $\bar{x}$, the affine-constrained prox is dual-representable: If y^* solves

$$0 \in A \ \mathrm{Prox}_{\lambda f}(z + A^* y) - b,$$

then $x^* := \mathrm{Prox}_{\lambda f}(z + A^* y^*)$ solves (P).

- Route: nonconvex primal $\rightarrow$ convex dual $\rightarrow$ primal recovery.

Numerical Experiments

- Machine: MATLAB R2023b on an Intel Xeon E5-2680 v3 workstation with 96 GB RAM.
- Tolerance: relative KKT residue: $= 10^{-6}$; Maxtime: $= 3600s$.

Solvers comparison:

Solver	LR Factorization	Handle Ineq
ManiSDP (Wang and Hu, 2025)	Yes	No
HALLaR/cuHALLaR (2025)	Yes	No
LoRADS/cuLoRADS (Han et al., 2024)	Yes	No
SDPF (Tang and Toh, 2024)	Yes	No
RiNNAL (Hou et al., 2025)	Yes	No
SketchyCGAL (Yurtsever et al., 2021)	No	Yes
SDPNAL+ (Yang et al., 2015)	No	Yes
SDPLR (Burer and Monteiro, 2003)	Yes	Yes
SDPDAL (Wang et al., 2023)	Yes	Yes
RiNNAL+ (ours)	Yes	Yes

Test Problems I

Binary integer nonconvex quadratic programming

$$\min\{x^\top Qx + 2c^\top x : \ x \in \{0,1\}^n\}. \tag{BIQ}$$

Maximum stable set problem

$$\max\{x^\top x : \ x_i x_j = 0 \ \forall (i,j) \in E, \ x \in \{0,1\}^n\}. \tag{θ_+}$$

Quadratic knapsack problem

$$\max\{x^\top Qx : \ a^\top x \leq \tau, \ x \in \{0,1\}^n\}. \tag{QKP}$$

Test Problems II

Cardinality-constrained minimum SOS clustering

$$\min\left\{\sum_{j=1}^{k}\frac{1}{c_j}\sum_{s,t=1}^{m}d_{st}\pi_j^{(s)}\pi_j^{(t)}:\ \pi\in\mathbb{S},\ \pi\in\{0,1\}^{m\times k}\right\},$$
$$\mathbb{S}:=\left\{\pi\in\mathbb{R}^{m\times k}:\ \sum_{j=1}^{k}\pi_i^{(j)}=1\ \forall i\in[m],\ \sum_{i=1}^{m}\pi_i^{(j)}=c_j\ \forall j\in[k]\right\}. \qquad \text{(ccMSSC)}$$

Sparse standard quadratic programming

$$\min\{x^\top Qx:\ e^\top x=1,\ e^\top u=\rho,\ x\le u,\ x\ge 0,\ u\in\{0,1\}^m\}. \qquad \text{(SStQP)}$$

Quadratic minimum spanning tree problem

$$\min\left\{x^\top Qx:\ x\in\{0,1\}^{|E|},\ e^\top x=|V|-1,\ \sum_{e\in E(S)}x_e\le|S|-1,\ \emptyset\neq S\subsetneq V\right\} \qquad \text{(QMSTP)}$$

Result Summary

Problem	Size	Rank ratio (%)	Speedup
(BIQ)	500–5000	3.5–10.0	77–187
(θ_+)	800–3000	1.0–28.1	40–106
(QKP)	500–5000	1.5–5.2	400
(ccMSSC)	720–2250	0.7–2.7	184–350
(SStQP)	200–1000	2.5–62.0	0.7–6
(QMSTP)	435–1225	1.1–41.6	0.3–23

- SStQP is further improved in the SC-SDP work.

RiNNAL-POP

Low-Rank ALM for Poly-SDP/Mom-SOS Relaxations of POP

Polynomial Optimization

Polynomial Optimization

$$\min_{x\in\mathbb{R}^n} \Big\{ f_0(x) \;\mid\; f_i(x) = 0,\ i \in [m],\ x \in D \Big\}. \tag{POP}$$

- $f_i(x)$: multivariate polynomials; D: semialgebraic domain.
- Inequalities can be encoded in D or by (squared) slack variables. (Ding and Wright, 2025)

Moment-SOS Relaxation

(Lasserre, 2001)

- Lower bound from SDP hierarchy.
- Asymptotically exact as order r increases.
- Var&Cons size grows rapidly: $\Omega(n^{2r})$.

Conic Reformulation

(Burer, 2009; Kim et al., 2020)

- Exact at a fixed-order τ.
- Untractable cone $\mathrm{co}(K)$.
- Lower bound by Poly-SDP relaxations.

RiNNAL-POP can solve both (Mom-SOS) and (Poly-SDP) relaxations.

Conic Reformulation

Convex Conic Reformulation (Kim et al., 2020)

$$\min_{X} \left\{ \langle Q^0, X \rangle \mid \langle Q^i, X \rangle = 0,\ i \in [m],\ X_{11} = 1,\ X \in \mathrm{co}(K) \right\}. \tag{COP}$$

- Monomial degree:

$$\tau \geq \max_{i=0,\ldots,m} \lceil \deg f_i / 2 \rceil .$$

- Monomial vector:

$$u_\tau(x) := (x^\alpha)_{|\alpha| \leq \tau} \in \mathbb{R}^{n_\tau},$$
$$n_\tau = \binom{n+\tau}{\tau}.$$

- Rank-one lift:

$$X := u_\tau(x) u_\tau(x)^\top \in \mathbb{S}^{n_\tau}.$$

- Lifted set:

$$K := \left\{ u_\tau(x) u_\tau(x)^\top : x \in D \right\}.$$

	1	x_1	x_2	$\cdots$	x^β
1	1	x_1	x_2	$\cdots$	x^β
x_1	x_1	x_1^2	x_1x_2	$\cdots$	x_1x^β
x_2	x_2	x_2x_1	x_2^2	$\cdots$	x_2x^β
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x^α	x^α	$x^\alpha x_1$	$x^\alpha x_2$	$\cdots$	$x^{\alpha+\beta}$

Tractable OA of $\mathrm{co}(K)$

Example of $\mathrm{co}(K)$

$$\tau = 1,\ D = \mathbb{R}^n \implies \mathrm{co}(K) = \mathbb{S}^{n+1}_+,$$
$$\tau = 1,\ D = \mathbb{R}^n_+ \implies \mathrm{co}(K) = \mathrm{CPP}^{n+1}.$$

Outer Approximation

$$\mathrm{co}(K) \subseteq \mathbb{S}^{n\tau}_+ \cap \mathcal{C} \cap \mathcal{P}$$

PSD cone ($\mathbb{S}^{n\tau}_+$)

$X \succeq 0.$

Consistency cone ($\mathcal{C}$)

$X_{\alpha,\beta} = X_{\gamma,\delta}$
if $\alpha + \beta = \gamma + \delta$

Polyhedral cone ($\mathcal{P}$)

Problem dependent,
E.g., $D = \mathbb{R}^n_+ \Rightarrow X \geq 0.$

This gives a tractable SDP relaxation and hence a lower bound for POP.

Identify Hidden Facial Structure

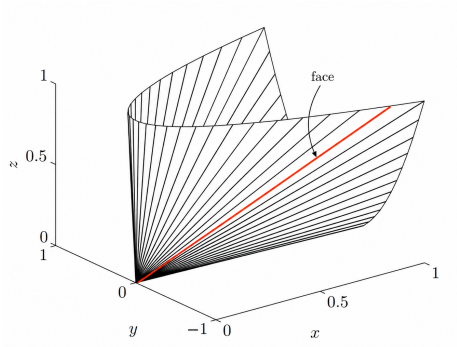
Key Insight. (COP) has many facial equalities.

Facial Constraints for $X \succeq 0$

$$AX = 0 \quad \Longleftrightarrow \quad \langle A^\top A, X \rangle = 0$$

Sources.

- RLT (Sherali, 2007)
- Moment-SOS (Lasserre, 2001)



Example

$$a^\top x = b \quad \Longrightarrow \quad \begin{bmatrix} -b & a^\top \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^\top = 0, \quad \Longrightarrow \quad \begin{bmatrix} -b & a^\top \end{bmatrix} X = 0.$$

Polyhedral-SDP Relaxation

Polyhedral-SDP Relaxation

$$\min_{X} \left\{ \langle Q^0, X \rangle \;\middle|\; \begin{array}{ll} \langle Q^i, X \rangle = 0, \ \forall i \in [\ell], \\ AX = 0, \quad X \in \mathbb{S}^{n_\tau}_+, \\ X_{11} = 1, \quad X \in \mathcal{P} \cap \mathcal{C}, \end{array} \right\}. \tag{Poly-SDP}$$

- A unified framework that includes SDP/DNN/RLT/Mom-SOS relaxations.

Theorem 5 (Relaxation Tightness)

(Informal) Poly-SDP relaxation is as tight as moment-SOS relaxation.

RiNNAL-POP for solving (Poly-SDP)

Compact Reformulation

$$\begin{aligned} \min_{X,Y} \quad & \langle Q^0, X\rangle + \delta_{\mathcal{X}}(X) + \delta_{\mathcal{Y}}(Y) \\ \text{s.t.} \quad & X - Y = 0, \quad \mathcal{Q}(X) = 0. \end{aligned} \tag{$\overline{\text{Poly-SDP}}$}$$

$\mathcal{X} := \{X \in \mathbb{S}^{n^\tau}_+ \mid AX = 0\}, \quad \mathcal{Y} := \{X \in \mathcal{P} \cap \mathcal{C} \mid X_{11} = 1\}, \quad \mathcal{Q}(X) := (\langle Q^i, X\rangle)_{i\in[\ell]}.$

Bottleneck	Scale	RiNNAL-POP idea
Nonnegativity constraints	$\Omega(n^{2\tau})$	fast projection
Consistency constraints	$\Omega(n^{2\tau})$	fast projection
PSD variable	$\Omega(n^{2\tau})$	low-rank factorization $X = RR^\top$
Facial equalities	$\Omega(mn^\tau)$	$AX = 0 \Longleftrightarrow AR = 0$
No Slater when $A \neq 0$	structural	work on the exposed face

Augmented Lagrangian Method

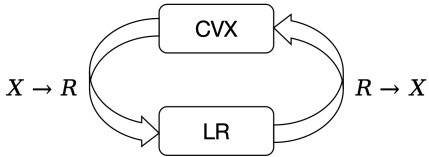
Augmented Lagrangian function.

$$\phi_k(X) = \langle Q^0, X\rangle + \frac{\sigma_k}{2}\|\mathcal{Q}(X) - \sigma_k^{-1}y^k\|^2 + \frac{\sigma_k}{2}\operatorname{dist}^2(X - \sigma_k^{-1}W^k, \mathcal{Y}).$$

ALM Iteration

$$\begin{aligned}
X^{k+1} &\approx \arg\min_{X\in\mathcal{X}} \ \phi_k(X),\\
Y^{k+1} &= \Pi_{\mathcal{Y}}(X^{k+1} - \sigma_k^{-1}W^k),\\
y^{k+1} &= y^k - \sigma_k\mathcal{Q}(X^{k+1}),\\
W^{k+1} &= W^k - \sigma_k(X^{k+1} - Y^{k+1}).
\end{aligned}$$

A Hybrid Method to Solve ALM Subproblem



CVX Phase

Projected-gradient step on X

$$\begin{aligned} \min_{X} \quad & \phi(X) \\ \text{s.t.} \quad & AX = 0,\ X \in \mathbb{S}^{n}_{+}{}^{r}. \end{aligned}$$

- Primal feasibility enforcement.
- Global convergence (Lee et al., 2023).
- Automatic rank selection.

LR Phase

Projected-gradient step on R

$$\begin{aligned} \min_{R} \quad & \phi(RR^{\top}) \\ \text{s.t.} \quad & AR = 0. \end{aligned}$$

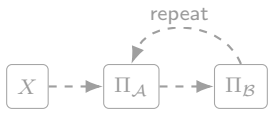
- Reduces variable dimension.
- Reduces the number of constraints.
- Reduces the objective value.

Exact Sequential Projection

Projection Subproblems

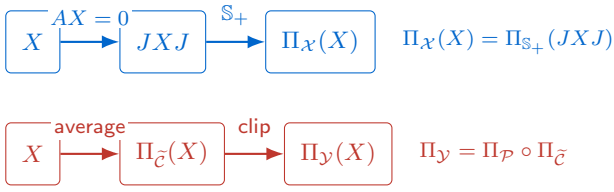
$$\mathcal{X} = \{X \in \mathbb{S}_+^{n_\tau} \mid AX = 0\}, \qquad \mathcal{Y} = \underbrace{\{X \in \mathbb{S}^{n_\tau} \mid X \geq 0\}}_{\mathcal{P}} \cap \underbrace{\{X \in \mathcal{C} \mid X_{11} = 1\}}_{\widetilde{\mathcal{C}}}.$$

Generic alternating projection



many passes
approximate projection

Our structure: one pass is exact



- Exact one-pass projection.
- For $\Pi_{\mathcal{Y}}$: complexity reduced from $\mathcal{O}(n^{7\tau})$ to linear $\mathcal{O}(n^{2\tau})$.

Simple Verification of Global Optimality

Assume (Poly-SDP) admits an optimal solution satisfying the KKT conditions

$$AX = 0, \quad \nabla\phi(X) - A^\top U - U^\top A = S,$$
$$\langle X, S\rangle = 0, \quad X \succeq 0, \quad S \succeq 0.$$

Theorem 6 (Recover dual variables)

A matrix $X \in \mathcal{X}$ is a minimizer of (Poly-SDP) *if and only if*

$$\widehat{S} := J(\nabla\phi(X))J \succeq 0$$
$$X\widehat{S} = 0$$

Numerical Experiments

- Machine: MATLAB R2023b on an Intel Xeon E5-2680 v3 workstation with 96 GB RAM.
- Tolerance: relative KKT residue: $= 10^{-6}$; Maxtime: $= 3600s$.
- Baseline solvers:
  1. RiNNAL-POP (ours)
  2. SDPNAL+ (Yang et al., 2015)
  3. GloptiPoly (Henrion et al., 2009)

Relaxation Comparison

Example: standard QP

$$\min \left\{ x^\top Q x : \; e^\top x = 1, \; x \in \mathbb{R}^n_+ \right\}.$$

Order	1st	2nd
Poly–SDP–RLT	0	0
Mom–SOS	∞	0

Table: Relative gaps (%).

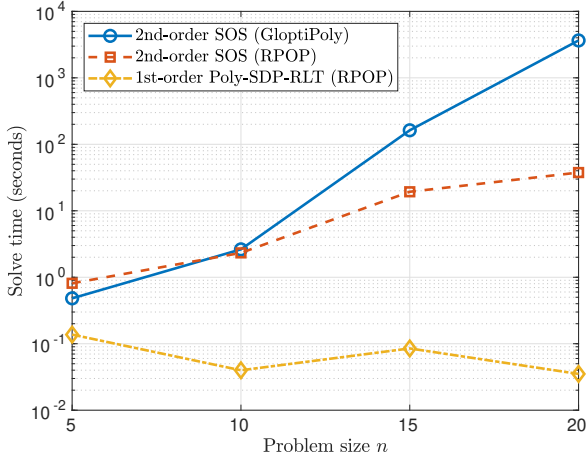


Figure: Computational time comparison.

Test Problems I

Standard quadratic programming

$$\min\left\{x^\top Qx:\ e^\top x=1,\ x\in\mathbb{R}^n_+\right\}. \tag{StQP}$$

Binary quadratic programs

$$\min\left\{x^\top Qx+c^\top x\mid x\in\{0,1\}^n\right\}. \tag{BIQ}$$

Minimum bisection problem

$$\min\left\{x^\top Lx:\ e^\top x=\frac{n}{2},\ x\in\{0,1\}^n\right\}. \tag{MBP}$$

Multiple quadratic knapsack problem

$$\max\left\{x^\top Q\,x+c^\top x\mid Ax\le b,\ x\in\{0,1\}^n\right\}. \tag{MQKP}$$

Test Problems II

Ball-constrained quartic minimization

$$\min\left\{\langle c, [x]_4\rangle \mid \|x\| \leq 1,\ x \in \mathbb{R}^{n-1}\right\}. \tag{BQM}$$

Kurtosis-minimization portfolio problem

$$\min\left\{\mathbb{E}_\xi\left[(x^\top(\xi-\mu))^4\right] \mid \mu^\top x = \mu_0,\ x^\top \Sigma x = \sigma_0^2,\ e^\top x = 1,\ x \geq 0\right\}. \tag{KMP}$$

Copositivity of symmetric tensors

$$\min\left\{\langle \mathcal{B}, \otimes^{\mathrm{t}} x\rangle \mid e^\top x = 1,\ x \geq 0\right\}. \tag{CST}$$

Best nonnegative rank-one tensor approximation problem

$$\min\left\{\|\mathcal{B} - \lambda x^{\otimes\alpha}\|^2 \mid \ \|x^{(i)}\| = 1,\ \lambda \geq 0,\ x^{(i)} \geq 0,\ \forall i \in [p]\right\}. \tag{NTF}$$

Result Summary

Problem	n	Deg	Relax	Eq	Ineq	≥ 0	Speedup
(StQP)	$1000 - 6000$	2	1	Simplex	-	✓	$34 - 36$
(StQP)	$20 - 80$	2	2	Simplex	-	✓	$3 - 6$
(BIQ)	$20 - 60$	2	2	Binary	-	✓	$10 - 25$
(MBP)	$30 - 60$	2	2	Equipartition	-	✓	$21 - 30$
(MQKP)	$20 - 40$	2	2	Binary	Knapsack	✓	$10 - 30$
(BQM)	$20 - 80$	4	2	-	Ball	-	$6 - 16$
(KMP)	$20 - 60$	4	2	Mean–variance	-	✓	$200 - 278$
(CST)	$1000 - 6000$	2	1	Simplex	-	✓	$18 - 500$
(CST)	$20 - 60$	4	2	Simplex	-	✓	$13 - 173$
(CST)	$5, 10$	8	4	Simplex	-	✓	$13 - 87$
(NSTF)	$20 - 80$	$3, 4$	2	Spherical	-	✓	$2 - 7$
(NTF)	$10 - 25$	$3, 4$	2	Block spherical	-	✓	$2 - 10$

RiNNAL-S

Low-Rank ALM for SC-SDP Relaxations of Sparse QP

SC-SDP Relaxation

Sparse Quadratic Program

$$\begin{aligned} \min_{x} \quad & x^\top Q x + 2c^\top x \\ \text{s.t.} \quad & Ax = b, \quad Bx \geq d, \quad \|x\|_0 \leq k. \end{aligned} \tag{SQP}$$

Theorem (Sparsity Cone)

$$\mathcal{K} := \left\{ Y \in \mathbb{S}^{n+1} \;\middle|\; \begin{bmatrix} kY_{11} & Y_{12} \\ Y_{21} & \operatorname{diag}(Y_{22}) \end{bmatrix} \succeq 0 \right\}.$$

$Y \in \mathcal{K}$ is exact for rank-one lifts: $Y = [1; x][1; x]^\top \Longrightarrow \|x\|_0 \leq k.$

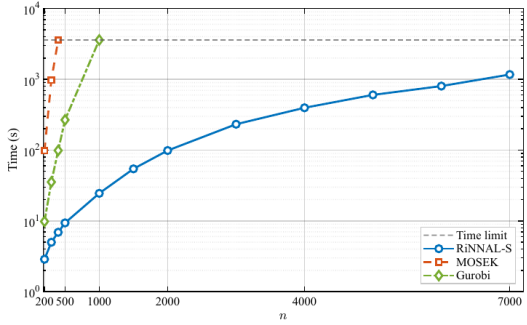
Sparsity-Cone Relaxation

$$\begin{aligned} \min_{Y} \quad & \langle \bar{Q}, Y \rangle \\ \text{s.t.} \quad & \mathcal{A}(Y) = 0, \quad \mathcal{B}(Y) \geq 0, \quad Y_{11} = 1, \\ & Y \in \mathcal{K} \cap \mathbb{S}^{n+1}_{+}. \end{aligned} \tag{SC-SDP}$$

- Same lower bound as SDP–RLT, but uses an $(n+1)$-dimensional matrix.

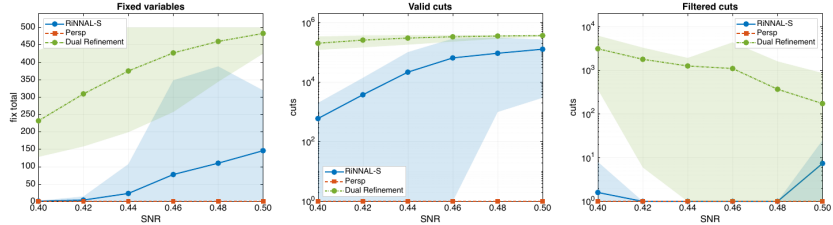
RiNNAL-S: Numerical Summary

Problem	Size	Speedup
SRR	200–7000	> 140
S-StQP	100–2000	30–700
S-BIQ	250–2500	34–77
S-QKP	500–3000	> 25



SC-SDP Presolving

Fix support from one root SC-SDP dual certificate.



(a) Fixed variables. (b) Valid cuts. (c) Filtered cuts.

(d) Solver time. (e) Relaxation gap. (f) Gurobi gap (600s).

References

1. Di Hou, Tianyun Tang, and Kim-Chuan Toh (2025). "A low-rank augmented Lagrangian method for doubly nonnegative relaxations of mixed-binary quadratic programs." *Operations Research* 74(3), 1626–1647.
2. Di Hou, Tianyun Tang, and Kim-Chuan Toh (2026). "RiNNAL+: a Riemannian ALM Solver for SDP-RLT Relaxations of Mixed-Binary Quadratic Programs." *Mathematical Programming Computation*, 1–57.
3. Di Hou, Tianyun Tang, and Kim-Chuan Toh (2025). "A Low-rank Augmented Lagrangian Method for Polyhedral-SDP and Moment-SOS Relaxations of Polynomial Optimization." *Mathematical Programming* (accepted), arXiv:2512.06359.
4. Di Hou, Thai P.D. Nguyen, Kim-Chuan Toh, and Guanyi Wang (2026). "Sparsity-Cone SDP Relaxations and Applications to Variable Fixing for Sparse Quadratic Programs." arXiv:2606.22894.
5. Di Hou, Tianyun Tang, Kim-Chuan Toh, and Shiwei Wang (2026). "On the efficient computation of proximal operators of affine-constrained nonconvex functions." arXiv:2602.23626.

Binary Integer Nonconvex Quadratic Programming

$$\min \left\{ x^\top Q x + 2c^\top x : \ x \le e, \ x \in \{0,1\}^n \right\}$$

Problem	Algorithm	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
$n = 500$	RiNNAL+	16, 850, 3	50	8.70e-07	-1.2259545e+05	8.0	0.9
	SDPNAL+	44, 97, 1535	76	9.98e-07	-1.2259531e+05	613.1	–
$n = 1000$	RiNNAL+	12, 650, 3	79	9.58e-07	-3.8983061e+05	19.3	2.3
	SDPNAL+	–	–	–	–	–	–
$n = 2500$	RiNNAL+	11, 600, 2	134	7.28e-07	-1.6096512e+06	133.3	15.9
	SDPNAL+	–	–	–	–	–	–
$n = 5000$	RiNNAL+	12, 650, 3	177	8.61e-07	-4.6838013e+06	1103.1	254.8
	SDPNAL+	–	–	–	–	–	–

- PG step is efficient due to preprocessing and warm-start.

Maximum Stable Set Problems

$$\max \left\{ x^\top x : \ x_i x_j = 0 \ \forall (i,j) \in E, \ x \leq e, \ x \in \{0,1\}^n \right\}$$

Problem	Algorithm	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
G11	RiNNAL+	56, 2850, 11	8	6.41e-07	-4.0000014e+02	47.7	6.4
$n = 800$	SDPNAL+	482, 722, 9358	2	2.46e-07	-4.0000005e+02	3008.3	–
G18	RiNNAL+	34, 1750, 7	75	9.08e-07	-2.7900052e+02	34.0	5.5
$n = 800$	SDPNAL+	–	–	–	–	–	–
G54	RiNNAL+	46, 2350, 10	168	9.01e-07	-3.4100141e+02	68.4	11.3
$n = 1000$	SDPNAL+	–	–	–	–	–	–
G30	RiNNAL+	13, 700, 3	98	9.00e-07	-5.7703872e+02	87.7	10.4
$n = 2000$	SDPNAL+	–	–	–	–	–	–
G50	RiNNAL+	24, 1250, 5	134	9.39e-07	-1.4940617e+03	754.8	149.3
$n = 3000$	SDPNAL+	–	–	–	–	–	–
1tc.1024	RiNNAL+	116, 5850, 24	288	9.38e-07	-2.0420520e+02	192.4	28.1
$n = 1024$	SDPNAL+	–	–	–	–	–	–
1tc.2048	RiNNAL+	94, 4750, 26	520	9.85e-07	-3.7049061e+02	820.7	162.8
$n = 2048$	SDPNAL+	–	–	–	–	–	–

- Efficient even on problems with high-rank solutions.

Quadratic Knapsack Problems

$$\max \left\{ x^\top Q x : \ a^\top x \le \tau, \ x \le e, \ x \in \{0,1\}^n \right\}$$

n	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
500	87, 4400, 18	26	9.91e-07	-1.1420448e+06	40.5	4.3
1000	27, 1400, 6	37	9.29e-07	-4.5986965e+06	53.1	5.5
2000	15, 800, 3	42	6.84e-07	-1.8316334e+07	123.7	22.7
5000	18, 950, 4	77	7.89e-07	-1.1399134e+08	1761.8	517.4

- SDPNAL+ fails to solve all instances.
- Projection and retraction nontrivial, but taking only 10–20% of total time.

Cardinality-Constrained Minimum SOS Clustering

$$\min \left\{ \sum_{j=1}^{k} \frac{1}{c_j} \sum_{s=1}^{m} \sum_{t=1}^{m} d_{st} \pi_j^{(s)} \pi_j^{(t)} \ : \ \pi \in \mathcal{S}, \ \pi \leq e_{m \times k}, \ \pi \in \{0,1\}^{m \times k} \right\}$$

where $\pi = [\pi^{(1)}, \ldots, \pi^{(k)}]$ with $\pi^{(j)} \in \{0,1\}^m$, and

$$\mathcal{S} := \left\{ \pi \in \mathbb{R}^{m \times k} : \sum_{j=1}^{k} \pi_i^{(j)} = 1 \ \forall i \in [m], \quad \sum_{i=1}^{m} \pi_i^{(j)} = c_j \ \forall j \in [k] \right\}$$

Problem	Algorithm	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
1	RiNNAL+	9, 500, 2	5	9.27e-07	7.3254609e+04	14.3	1.4
$n = 720$	SDPNAL+	-	-	-	-	-	-
2	RiNNAL+	3, 200, 1	7	8.78e-07	4.0536895e+09	12.6	1.1
$n = 902$	SDPNAL+	54, 246, 2312	1	7.91e-07	4.0536886e+09	2313.4	-
3	RiNNAL+	12, 511, 3	21	3.90e-07	8.5151301e+02	1500.3	193.9
$n = 1662$	SDPNAL+	-	-	-	-	-	-
4	RiNNAL+	2, 150, 1	28	5.87e-07	7.3610392e+08	42.2	13.7
$n = 1800$	SDPNAL+	-	-	-	-	-	-
5	RiNNAL+	51, 2600, 11	61	5.42e-07	1.0500893e+06	2215.9	170.8
$n = 2250$	SDPNAL+	-	-	-	-	-	-

- Efficient for many constraints (up to 3.7M eq/ineq and 5M nonneg).

Sparse StQP

$$\min\left\{x^\top Qx:\ e^\top x=1,\ \|x\|_0\le\rho,\ x\in\mathbb{R}^m_+\right\},$$

Big-M reformulation:

$$\min\left\{x^\top Qx:\ e^\top x=1,\ e^\top u=\rho,\ x\le u\le e,\ x\in\mathbb{R}^m_+,\ u\in\{0,1\}^m\right\}.$$

Problem	Algorithm	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
COP	RiNNAL+	125, 6300, 25	102	6.64e-07	-1.0190589e-01	30.7	2.1
$n=200$	SDPNAL+	76, 112, 1170	102	8.59e-07	-1.0196083e-01	40.1	-
PSD	RiNNAL+	374, 18750, 75	5	8.59e-07	1.4174794e-02	80.7	14.5
$n=200$	SDPNAL+	801, 1058, 17306	2	8.50e-07	1.4174999e-02	468.1	-
SPN	RiNNAL+	747, 37400, 150	107	9.31e-07	1.3592384e-02	204.0	20.8
$n=200$	SDPNAL+	801, 1032, 29187	12	9.94e-07	1.4887842e-02	866.1	-
COP	RiNNAL+	300, 15050, 60	202	9.99e-07	-1.0402158e-01	230.5	15.8
$n=400$	SDPNAL+	103, 145, 1511	202	9.24e-07	-1.0445084e-01	170.7	-
PSD	RiNNAL+	360, 18050, 72	108	9.50e-07	9.0349366e-03	231.0	19.4
$n=400$	SDPNAL+	-	-	-	-	-	-
SPN	RiNNAL+	1812, 90650, 363	248	9.97e-07	-1.2530973e-02	1556.1	145.7
$n=400$	SDPNAL+	-	-	-	-	-	-
PSD	RiNNAL+	507, 25400, 102	234	9.88e-07	2.9188026e-03	1838.0	152.6
$n=1000$	SDPNAL+	-	-	-	-	-	-

Quadratic Minimum Spanning Tree Problem

$$\min \left\{ x^\top Q x : x \in \mathcal{T}' \right\}.$$

where

$$\mathcal{T}' := \left\{ x \in \{0,1\}^m : x \le e,\ \sum_{e \in E} x_e = n-1,\ \sum_{e \in \partial S} x_e \ge 1,\ \forall S \subsetneq V,\ |S| = 1 \right\},$$

Problem	Algorithm	Iteration	Rank	$\mathrm{R}_{\max}$	Objective	Time	TPG
vsym	RiNNAL+	29, 1480, 6	5	4.86e-07	7.7804931e+04	12.9	1.7
$m = 435$	SDPNAL+	148, 230, 3713	1	2.91e-07	7.7804694e+04	299.7	-
sym	RiNNAL+	103, 5190, 21	197	7.70e-07	5.3262719e+03	99.3	2.8
$m = 435$	SDPNAL+	96, 109, 1486	195	1.09e-06	5.3262716e+03	130.5	-
esym	RiNNAL+	4912, 254289, 4912	49	9.85e-07	7.4273402e+03	3456.3	931.4
$m = 435$	SDPNAL+	529, 692, 10111	46	9.55e-07	7.4273883e+03	1083.3	-
vsym	RiNNAL+	96, 4850, 22	8	7.99e-07	1.6444786e+05	289.2	64.3
$m = 1225$	SDPNAL+	-	-	-	-	-	-
sym	RiNNAL+	84, 4275, 17	509	8.07e-07	1.4104759e+04	484.0	21.2
$m = 1225$	SDPNAL+	192, 199, 2628	507	1.22e-06	1.4104759e+04	1987.4	-

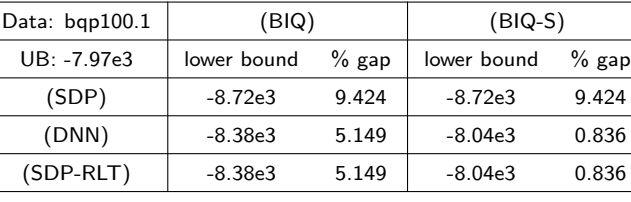
Tightening Strategy

$$\min \left\{ x^\top Q x + 2c^\top x : \ x \in \{0,1\}^n \right\} \qquad \text{(BIQ)}$$

Strengthen by adding redundant bound constraint:

$$\min \left\{ x^\top Q x + 2c^\top x : \ x \leq e, \ x \in \{0,1\}^n \right\} \qquad \text{(BIQ-S)}$$

Data: bqp100.1	(BIQ)		(BIQ-S)	
UB: -7.97e3	lower bound	% gap	lower bound	% gap
(SDP)	-8.72e3	9.424	-8.72e3	9.424
(DNN)	-8.38e3	5.149	-8.04e3	0.836
(SDP-RLT)	-8.38e3	5.149	-8.04e3	0.836



Time Comparison

- Solving SDP-RLT is much faster due to lower dimensionality.

